Category Theory in the hands of physicists, mathematicians, and philosophers

Category Theory in Physics, Mathematics, and Philosophy, Kuś M., Skowron B. (eds.), Springer Proc. Phys. 235, 2019, pp.xii+134.

The book under review is a result of the conference under the same title which was held in Warsaw, Poland in November 2017. As far as I know, it was the first conference in Poland's history on the subject of category theory (CT) and its applications. Warsaw is an excellent place for such an event if only because of its traditions of the Lvov–Warsaw School, but also being the place where Samuel Eilenberg (one of the founding fathers of CT) was born and defended his Ph.D. thesis. The majority of the organizers and speakers were Polish. Nevertheless the significance and impact of this conference, mainly thanks to the proceedings, is definitely international. The present volume of ZFN, which itself is devoted to the conference proceedings of the next great event of this kind, shows that the tradition of Polish conferences on the subject of CT and its applications emerges. It could only be wished it would thrive in the future.

The publication consists of 10 chapters, which are all separate articles, the majority of which are papers presented at the conference. The subjects of the papers are very diverse, which itself evinces how vast are the applications of CT to different areas of knowledge.[1] The reader not familiar with at least the basics of CT will be much hindered in the study of them.[2] However, not all the papers hinge on the technicalities of CT (see the first four chapters). Others also require advanced knowledge (mainly of physics, for chapters 7 to 9).

The book begins with a few pages of introduction, which presents the work's context and succinctly discusses each chapter. The first article, *Why Categories?*, written by M. Kuś, B. Skowron, and

---

[1] The reader interested in the subject of broad applications of CT may benefit also from e.g. the great book (Awodey, 2010) (ZFN published its Polish review by Michael Heller (2018)).

[2] Textbooks in CT that may be consulted are e.g. (Awodey, 2010; Smith, 2018; McLarty, 1995).

K. Wójtowicz, is also a kind of intro-
duction to the whole book, but from
a different perspective. It poses a
question (and attempts to answer it,
at least partially) about reasons for
such a significant increase in pop-
ularity of CT and its applications
in recent years. The great diversity
of the following chapters is a good,
though still only partial, example
of this phenomenon. This paper
presents some aspects of the origins
of CT, its relations with, and possi-
ble applications to philosophy and
physics. Within applications to phi-
losophy, it discusses structuralism
in the philosophy of mathematics,
foundations of mathematics, unity
of mathematics, and metaphysics.
In connection with physics, it ex-
amines mainly quantum mechan-
ics and its ontological foundations.
Nonetheless, the text is more than
a review. The authors, when dis-
cussing the above mentioned dif-
ferent applications of CT, also give
examples of their main claim: "cat-
egory theory is a formal ontology
that captures the relational aspects
of the given domain in question"
(p.1).[3] These examples show how
the formal-ontological shift brought
by CT, meant as the shift from the

"standard and natural attitude to-
wards the objects, as if they were
individual subjects of properties
[. . .] [to] a form of pure relational-
ity" (p.18), sheds some new light on
the old problems.

The next chapter, *Category The-
ory and Philosophy*, by Z. Król,
deals with relations between CT
and philosophy. One of the topics
addressed in this article, although
not the main one, is how set the-
ory (ST) and CT are similar and dif-
ferent. The author considers them
more broadly than just the formal
theories: "CT and ST are not singu-
lar formal theories, but rather open
domains accompanied by the rel-
evant methods and styles of con-
sideration, together with some ba-
sic concepts which can be investi-
gated within many different formal
theories" (p.22). The main part of
this article is a case study of cer-
tain classical problems in philoso-
phy and the question of the useful-
ness of CT in dealing with them.
The study is rather general without
deeper analysis, albeit the author
gives some details and references to
literature. It starts with remarks on
the great influence of mathematics—
in general—upon philosophy in its

---

[3] References with no information except the page number refer to the book under
review.

history. The studied cases concern some issues in ontology (such as monism vs pluralism, or mathematical Platonism) and epistemology.

K. Wójtowicz in his paper, *Are There Category-Theoretical Explanations of Physical Phenomena?* (chapter 3), addresses a question of mathematical explanation in science. He asks especially whether CT may give explanations of physical phenomena. To answer this question he first analyzes the general problem of the explanatory role of mathematics in physics, assuming (as a working hypothesis) that "there are genuine mathematical explanations in science" (p.37). He admits that "it cannot be denied that CT contributes to our understanding of physics" (p.33) and gives some illustrations for this, mainly within the Topos Quantum Theory. However, after noting first that "if a model has no predictive potential at all, it is doubtful whether it really has explanatory character" (p.38), his central claim is that "in this sense there are no category-theoretical explanations of physical phenomena, in spite of there being mathematical explanations" (p.38). The author considers the contribution of CT to be "on the metatheoretical rather than the theoretical level" (p.40). He eventually writes: "CT offers 'meta-abstract explanations'"

(p.42). I have the feeling that this article underestimates rather than overestimates even the hitherto relevance of CT to other disciplines. Besides, maybe the future will show more.

Chapter 4, entitled *The Application of Category Theory to Epistemic and Poietic Processes*, by J. Lubacz, tries to explore the possibility of applying CT and its notional framework to "the analysis and monitoring of progress in the unfolding of [. . .] processes," (p.45) such as acquiring knowledge and some activity that results in the creation of artefacts. These are called epistemic and poietic processes, respectively. The style of this paper is definitely philosophical. After the introduction, the author presents some considerations about the processes in question in the broader context of philosophy as well as presenting his own proposition of the possible "conceptual structure of the pattern and dynamicity of epistemic and poietic processes" (p.48). Next, the author ponders the potential application of CT to epistemic and poietic processes. He rightly notes that "it must be clearly stated that the apparatus of CT can only be employed for those conceptual components of epistemic and poietic processes which are expressible in some for-

mal language and form" (p.50). He then develops his own proposal of the possible use of CT in this context, which is, however, rather general and preliminary in nature.

In his paper, *Asymmetry of Cantorian Mathematics from a Categorial Standpoint: Is It Related to the Direction of Time?* (chapter 5), Z. Semadeni addresses an interesting feature of the *Cantorian Mathematics*. In his approach, *Cantorian Mathematics* is referring to "basic mathematical structures of algebra, topology, functional analysis etc. expressed in terms of set theory, as they were conceived prior to the emergence of category theory, i.e., by the middle of the 20th century" (p.55f). The author notes and explains briefly that CT, in a certain sense (with respect to the reversal of arrows), is symmetric. However, when we consider some categories with objects being structures taken from *Cantorian Mathematics*, then the products and coproducts of these objects (separately in each category) each have a different 'style.' Namely, while almost all examples considered (and the author gives quite a few of them) of products turn out to be "the cartesian product endowed with a suitable structure" (p.57) (and the remaining two

examples can be in some way cured or revised alike), coproducts, however, "may be markedly different from each other" (p.57). For some categories coproducts "are based on the same construction, namely on the disjoint union" (p.57), but "in other categories coproducts may differ basically" (p.57). As an example, one of many the author gives, let me note the category **Grp** (of groups and their homomorphisms) in which the coproduct is the free product of groups, whereas in its full subcategory of abelian groups the coproduct is the (external) direct sum of groups. The author notes that similar asymmetry concerns also equalizers and coequalizers ("albeit in a much milder form" (p.59)), and other limits and colimits. At this point, he poses a philosophical question: "What features of *Cantorian Mathematics* lie behind this asymmetry?" (p.60), and suggests that it follows "from the asymmetry of many-to-one relationship in the notion of a function $f : X \rightarrow Y$" (p.60). In the last two paragraphs, we find an interesting discussion showing that the asymmetry between products and coproducts changes, so to speak, direction, when instead of considering examples of an "algebraic" nature

one considers structures of "coalgebraic" nature connected with one-to-many relation ("mostly stimulated by Computer Science" (p.61)).

The next article, entitled *Extending List's Levels*, by N. Dewar, S.C. Fletcher, and L. Hudetz is an interesting extension and modification of the unified framework for modeling different types of levels (descriptive, explanatory and ontological) proposed by C. List (2019). The paper is well written—particular notions are clearly introduced and commented on and occasionally important and helpful examples are given. First, the authors succinctly review List's approach and correct a minor defect. Next, they analyze the relationship between supervenience and reduction in this setting. In general, supervenience does not entail reduction (and the reader is familiarized through a simple example), but the authors have shown that if the levels and supervenience maps fulfill certain additional conditions (they have to be compatible and jointly characterizable, the notions being defined and commented in the text), then supervenience does entail reduction. They note, moreover, that "in many cases of supervenience between scientific levels of description, this [fulfilling the above-mentioned condition(s)] can be expected. So it is quite plausible that in many cases of interest, supervenience and reduction of levels go hand in hand" (p.79). After these considerations, the authors propose two extensions of List's framework: from supervenience maps treated as (total) functions to partial maps, and from surjective maps to non-surjective ones. These generalizations open new possibilities, described in the text. Subsequently, the authors move on to the most important, in my opinion, part of their work, namely to the modification of List's framework which involves considering levels not only as elements of a certain poset (or objects of a posetal category) but as categories of structures (more precisely, they suggest "to represent a level of description, L, as a pair $\langle \mathbf{L}, \Omega \rangle$ consisting of a description language, $\mathbf{L}$, and a category, Ω, of $\mathbf{L}$-structures" (p.73)). This means that in order to specify a level of description "one does not only specify its structures but also the morphisms (admissible transformations) between these structures" (p.73). The choice of morphisms "reflects which expressions of $\mathbf{L}$ are taken to be meaningful within the level L" (p.73). Moreover, in this setting superve-

nience relations between the levels are viewed as functors. Another extension of the original framework is made by allowing "all sorts of functors to be included in a system of levels of description" (p.77), rather than just the supervenience functors. The authors give also some examples involving these generalizations and note that the extended framework better serves for a philosophy of science in general. This last generalization (of treating levels as categories) is far-reaching, as "taking levels as categories themselves demands a more robust use of categorial ideas [such as natural transformations, adjunctions, and others] that could also prove to be more fruitful" (p.79).

The next three chapters deal with applications of CT to physics. The first of them, written by K. Bielas and J. Król, titled *From Quantum-Mechanical Lattice of Projections to Smooth Structure of* $\mathbb{R}^4$, uses CT to relate quantum algebra structure with the smooth structure of spacetime. The paper is a work in progress and is connected with some earlier article by the authors and another scholar, namely Król et al. (2017). After the introduction, in which the authors already bring in some key concepts,

they outline some quantum mechanical preliminaries, i.a. introducing a complete orthomodular lattice of projections on a Hilbert space associated with the initial quantum system. This lattice (denoted as $\mathbb{L}$) is an algebraic basis of the so-called quantum logic. As is well known, generally (whenever dim $\mathscr{H} > 2$) $\mathbb{L}$ is not Boolean, which means that logic of a quantum system defined in this way is not classical. In order to get the connection with the classical world the authors, in the next section, look at the subalgebras of $\mathbb{L}$ which are Boolean. Each such Boolean subalgebra, they argue, is in a certain way "to be considered as a local, classical frame of reference for a quantum system" (p.87). Subsequently, they show a way to construct an orthomodular lattice from its Boolean subalgebras as a suitable colimit in the category of so-called partial Boolean algebras and appropriate homomorphisms (one has to do it in this larger category, which extends the category of orthomodular lattices and lattice homomorphisms). Then it is shown, in the next section, how one can arrive at the smooth manifold by means of some categorical constructions, namely the authors conclude that "given a smooth manifold, it is al-

ways the colimit of its atlas" (p.89).[4]
Finally, the authors try to relate the
quantum structure with the differential (smooth) structure of the manifold. However, it is still an open
problem if certain correspondence
has a functorial character. In the *Discussion* section some further considerations are addressed, i.a. the cardinality of the smooth atlas of exotic
R^4.

Differential structure of a manifold, or rather its enrichment in
terms of a so-called formal manifold is a key tool of the next paper,
Beyond the Space-Time Boundary,
by M. Heller and J. Król. The article is quite technical, preparatory
for further research, but also fundamental and opening a truly intriguing path for new explorations.[5] In
order to "cross the boundary" of a
singular spacetime, they use the so-
called Synthetic Differential Geometry (SDG), which is a categorical
version of standard differential geometry and is based on intuition-

istic logic. After the introduction,
the authors offer many new notions
(known in the literature) needed
for further considerations and their
model in the next sections. Among
others, one considers various kinds
of infinitesimals, which enrich the
standard structure of $\mathbb{R}$ (or a manifold in general) and make differentiation a purely algebraic operation. The authors give a nice image:
"We may imagine that they [infinitesimals] constitute the entire world inside every point of $\mathbb{R}$, a sort of a
fiber over $x \in \mathbb{R}$" (p.96). The reader
not familiar with such infinitesimals may find the definition such
as "$D = \{x \in R | x^2 = 0\}$" (p.97)
astonishing. Let me only note that
the intuitionistic logic plays one of
the key roles here. In SDG such an
object D obviously does not reduce
to $\{0\}$. It comprises so-called nilsquare infinitesimals, which are different, but at the same time indistinguishable, from 0.[6] The whole collection of (countably many) differ-

---

[4] The reader might find it helpful to note that there is a certain confusion with the
notation: on p.88, V_i is a subset of $\mathbb{R}^n$, whereas in the Corollary 2, on p.89, V_i is a
subset of the manifold. It would be more natural to denote on p.89, in accordance with
the notation from p.88, V_i as U_i, and W_i as V_i.

[5] The interested reader may consult also other papers in this subject by these authors,
e.g. see (Heller and Król, 2016; 2017).

[6] In the intuitionistic logic, double negation does not, in general, imply identity (double
negation elimination is not a theorem), so not being different from 0 (not being not
equal 0) does not imply being identical to zero or, in other words, indistinguishability
does not, in general, imply identity.

ent kinds of infinitesimals serves to distinguish various kinds of neighbourhoods, which in turn allow us to define various kinds of the so-called monads. In the proposed model, for example when thinking about the evolution of the universe back in time, these monads, which are ordered (relative to each other) and represent various kinds of differentiability properties, provide suitable structures to describe the evolution after crossing the singular boundary (which from the physical point of view might possibly be identified with Planck's threshold). The authors note that the model "does not pretend to describe the actual evolution of the universe. At its present stage of development, it is nothing more than a toy model" (p.96). The toy models, however, can play a surprisingly significant role in the development of more realistic physical models! Two appendices give more details, for example about the way infinitesimal spaces may appear in the model.

The methods of SDG are also the main mathematical tools used in the next paper in application to the pursuit of quantum gravity (QG). J. Król in his article *Aspects of Perturbative Quantum Gravity on Synthetic Spacetimes* gives his own con-tribution to this huge endeavor. In the introduction, the author presents some of the results in QG obtained so far. He follows the known perturbative approach to QG (which involves an expansion of the metric tensor around the flat Minkowski spacetime and an attempt to quantize the fluctuations around it) but tries to use new methods in the context of SDG. As he notes, "The particularly well-defined area of applicability of synthetic methods will appear to be perturbative quantum gravity" (p.107). Some definitions and notions are common with the previous chapter (which J. Król co-authored). The reader has to be familiar with some background knowledge and notations from the literature in order to fully follow the contents of the paper. I will not bring up the details of the article, which comprises much more considerations than are mentioned here. I only note that the author considers both the case of pure gravity (no matter fields) and the case with matter. Synthetic methods are applied in the context of the so-called BCFW procedure (from the names of the authors of (Britto et al., 2005), also see Appendix B of the paper by Król) and lead, *inter alia*, to the following results: (i) "The pure perturbative

covariant QG on synthetic space-time is entirely finite theory [. . .] [and] $g_{\mu\nu}$ can now be considered as fully quantized field" (p.112); (ii) "supergravity theory [for $\mathcal{N} = 8$ and with matter fields] formulated on synthetic spacetime is again finite and $g_{\mu\nu}$ quantized" (p.112). In the last section the author also reflects on the question, "how (if at all) it can be that a perturbative QG is considered a fundamental theory" (p.113).

The last chapter, titled *Category Theory as a Foundation for the Concept Analysis of Complex Systems and Time Series*, by G.N. Nop, A.B. Romanowska and J.D.H. Smith, deals with the applications of CT to so-called concept analysis. The paper extends and generalizes the existing applications. Concept analysis studies relationships between some items and properties they possess. After the short introduction, the authors give a summary of the notions used in this context. Although they are adopted in various fields of study (and may have different names), they "provide essentially equivalent tools for the analysis of a static system functioning at a single level" (p.120). To get some taste of the paper let me only mention one special notion, that of

a concept of a certain context. A context is a triple made of a set of items (Ω), a set of properties (Π), and a relation between them attributing properties to items (being a relation it is a subset of $\Omega \times \Pi$). A concept of such a context is an ordered pair (A, B) (denoted in the text as $(A|B)$), where A is a subset of items and B is a subset of properties, such that the set of properties common to all items in A is exactly B and at the same time the set of items attributed to all properties in B is exactly A. I think one may say that in this way we get a certain characterization of some items in terms of properties (and vice versa). In general, the power sets of Ω and Π (treated as poset categories) are connected with certain adjoint functors known as a Galois connection. In the paper, the reader is familiarized with many other notions and some simple examples as well. Then the authors proceed to "present the foundations for an extension of the ideas of concept analysis to changing environments, evolving complex systems, and time series analyses of successive stages of a given system" (p.119), which is the main goal of the paper. The extension to complex systems is done, generally speaking, in terms of certain functors from a

semilattice (which serves as a kind of index of distinct levels or, if it is a chain, of time points) to appropriate categories, though the procedure is more sophisticated.

In sum, the book under review is a testimony to the breadth of applications of CT in physics, mathematics, and philosophy. The book is worth reading, in general, although the chapters are so diverse (in the subject matter and sometimes also in the scientific value) that I suppose the reader that is not interested in all of the interdisciplinary applications should focus on the part (s)he is interested in; e.g. chapters 7–9 may be truly engaging for readers familiar with general relativity and quantum mechanics but may be completely incomprehensible for non-physicists. The papers contained in the book are not review articles nor popular science, but are mostly conference papers, which makes the reading much more demanding (recommended for graduate-level readers and beyond) though at the same time much more fascinating.

Mariusz Stopa

Awodey, S., 2010. *Category Theory*. 2nd ed., *Oxford Logic Guides* 52. Oxford; New York: Oxford University Press.

Britto, R., Cachazo, F., Feng, B. and Witten, E., 2005. Direct proof of tree-level recursion relation in Yang-Mills theory. *Phys. Rev. Lett.* [Online], 94, p.181602. Available at: arXiv: hep - th / 0501052. Available at: https://doi.org/10.1103/PhysRevLett.94.181602.

Heller, M. and Król, J., 2016. How Logic Interacts with Geometry: Infinitesimal Curvature of Categorical Spaces. Available at: arXiv: 1605.03099 [math.DG].

Heller, M. and Król, J., 2017. Infinitesimal Structure of Singularities. *Universe* [Online], 3(1), p.16. Available at: https://doi.org/10.3390/universe3010016.

Heller, M., 2018. Filozoficznie prowokująca teoria kategorii. *Philosophical Problems in Science (Zagadnienia Filozoficzne w Nauce)* [Online], (65), pp.232–241. Available at: https://zfn.edu.pl/index.php/zfn/article/view/451 [visited on 24 July 2020].

Król, J., Asselmeyer-Maluga, T., Bielas, K. and Klimasara, P., 2017. From Quantum to Cosmological Regime. The Role of Forcing and Exotic 4-Smoothness. *Universe* [Online], 3(2), p.31. Available at: https://doi.org/10.3390/universe3020031.

List, C., 2019. Levels: Descriptive, Explanatory, and Ontological.

Noûs [Online], 53(4), pp.852–883. Available at: https://doi.org/10.1111/nous.12241.

McLarty, C., 1995. *Elementary Categories, Elementary Toposes*. Oxford: Oxford University Press.

Smith, P., 2018. *Category Theory: A Gentle Introduction* [Online]. University of Cambridge, lecture notes. Available at: http://www.logicmatters.net/resources/pdfs/GentleIntro.pdf [visited on 24 July 2020].