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POSSIBLE PHYSICAL UNIVERSES

1. INTRODUCTION

Einstein famously claimed that "what really interests me is whether God had any choice in the creation of the world." This is generally considered to be a whimsical version of the question, 'is there only one logically possible physical universe?' The modern answer to this question is: 'apparently not!'

In this paper, we will approach the notion of possible physical universes using the philosophical doctrine of structural realism, which asserts that, in mathematical physics at least, the physical domain of a true theory is an instance of a mathematical structure.[1] It follows that if the domain of a true theory extends to the entire physical universe, then the entire universe is an instance of a mathematical structure. Equivalently, it is asserted that the physical universe is isomorphic to a mathematical structure. Let us refer to this proposal as 'universal structural realism'.

Whilst the definition of structural realism is most frequently expressed in terms of the set–theoretical, Bourbaki notion of a species of mathematical structure, one can reformulate the definition in terms of other approaches to the foundations of mathematics, such as mathematical category theory. In the latter case,

[1]This notion was originally advocated by Patrick Suppes (1969), Joseph Sneed (1971), Frederick Suppe (1989), and others.

one would assert that our physical universe is an object in a mathematical category.

Many of the authors writing about such ideas neglect to assert that the universe is *an instance* of a mathematical structure, but instead claim that the universe *is* a mathematical structure. In some cases this can be taken as merely an abbreviated form of speech; in others, the distinction in meaning is deliberate, and such authors may be sympathetic to the notion that the physical universe is nothing but form, and the notion of substance has no meaning. This corresponds to Ladyman's distinction (1998) between the ontic and epistemic versions of structural realism. Whilst the epistemic version accepts that mathematical physics captures the structure possessed by the physical world, it holds that there is more to the physical world beyond the structure that is possesses. In contrast, the ontic version holds that the structure of the physical world is the only thing which exists.

Those expressions of universal structural realism which state that 'the' physical universe is an instance of a mathematical structure, tacitly assume that our physical universe is the only physical universe. If one removes this assumption, then universal structural realism can be taken as the two–fold claim that (i) our physical universe is an instance of a mathematical structure, and (ii) other physical universes, if they exist, are either different instances of the same mathematical structure, or instances of different mathematical structures. Given that mathematical structures are arranged in tree–like hierarchies, other physical universes may be instances of mathematical structures which are sibling to the structure possessed by our universe. In other words, the mathematical structures possessed by other physical universes may all share a common parent structure, from which they are derived by virtue of satisfying additional conditions. This would enable us to infer the mathematical structure of other physical universes by first generalizing from the mathematical structure of our own,

and then classifying all the possible specializations of the common, generic structure.

It is common these days to refer to the hypothetical collection of all physical universes as the multiverse. I will refrain from using the phrase 'ensemble of all universes', because an ensemble is typically considered to be a space which possesses a probability measure, and it is debatable whether the universe collections considered in this paper possess a natural probability measure. It should also be emphasised that no physical process will be suggested to account for the existence of the multiverses considered in this paper. This paper will not address those theories, such as Linde's chaotic inflation theory (1983a and 1983b) or Smolin's theory of cosmological natural selection (1997), which propose physical processes that yield collections of universes or universe–domains. The universes considered in this paper are mutually disjoint, and are not assumed to be the outcome of a common process, or the outcome of any process at all.

If a physical universe is conceived to be an instance of a mathematical structure, i.e. a structured set, then it is natural to suggest that the multiverse is a collection of such instances, or a collection of mathematical structures. Tegmark, for example, characterises this view as suggesting that "some subset of all mathematical structures... is endowed with... physical existence," (1998, p1). As Tegmark points out, such a view of the multiverse fails to explain why some particular collection of mathematical structures is endowed with physical existence rather than another. His response is to suggest that all mathematical structures have physical existence.

Like many authors, Tegmark implicitly assumes that the physical universe must be a structured set (or an instance thereof). As Rucker asserts, "if reality is physics, if physics is mathematics, and if mathematics is set theory, then everything is a set," (Rucker 1982, p200). However, as already alluded to, mathematics cannot be identified with set theory, hence it doesn't follow that every-

thing is a set. There are mathematical objects, such as topoi, which are not sets. Hence, it may be that the physical universe is not a set.

Category theory is able to embrace objects which are not sets. A category consists of a collection of objects such that any pair of objects has a collection of morphisms between them. The morphisms satisfy a binary operation called composition, which is associative, and each object has a morphism onto itself called the identity morphism. For example, the category **Set** contains all sets as objects and the functions between sets as morphisms; the category of topological spaces has continuous functions as morphisms; and the category of smooth manifolds has smooth (infinitely–differentiable) maps as morphisms. One also has categories such as the category **Top** of all topoi, in which the morphisms need not be special types of functions and the objects need not be special types of sets. One can therefore generalize the central proposition of universal structural realism, to assert that the physical universe is an object in a mathematical category.

If the physical universe is not a structured set of some kind, then the multiverse may not be a collection of structured sets either. In fact, in the case of Tegmark's suggestion that the multiverse consists of all mathematical sets, it is well–known that there is no such thing as the set of all sets, so Tegmark would be forced into conceding that the multiverse is not a set itself, but the category of all sets, perhaps. More generally, one could suggest that the multiverse is a collection of mathematical objects, which may or may not be structured sets. The multiverse may be a category, but it may not be a category of structured sets. The analogue of Tegmark's suggestion here would perhaps be to propose that all categories physically exist. Categories can be related by maps called 'functors', which map the objects in one category to the objects in another, and which map the morphisms in one category to the morphisms in the other category, in a way which preserves the composition of morphisms. Furthermore, one

can relate one functor to another by something called a 'natural transformation'. In effect, one treats functors as higher–level objects, and natural transformations are higher–level morphisms between these higher–level objects. Accordingly, natural transformations are referred to as '2–morphisms'. Whilst a category just possesses objects and morphisms, a 2–category possesses objects, morphisms and 2–morphisms. The collection of all categories is a 2–category in the sense that it contains categories, functors between categories, and natural transformations between functors; each category is an object, the functors between categories are morphisms, and the natural transformations are 2–morphisms. However, the 2–category of all categories is only one example of a 2–category. A 2–category which has categories as objects, need not contain all categories, and a 2–category need not even have categories as objects. To fully develop Tegmark's suggestion, one would need to propose that all 2–categories physically exist, and the latter itself is just one example of a 3–category. One would need to continue indefinitely, with the category of all n–categories always just one example of an $n+1$–category, for all $n \in \mathbb{N}$.

Let us return, however, to the notion that a physical universe is an instance of a structured set, and let us refine this notion in terms of mathematical logic. A theory is a set of sentences, in some language, which is closed under logical implication. A model for a set of sentences is an interpretation of the langauge in which those sentences are expressed, which renders each sentence as true. Each theory in mathematical physics has a class of models associated with it. As Earman puts it, "a practitioner of mathematical physics is concerned with a certain mathematical structure and an associated set $\mathfrak{M}$ of models with this structure. The... laws L of physics pick out a distinguished sub–class of models $\mathfrak{M}_L := \mathrm{Mod}(L) \subset \mathfrak{M}$, the models satisfying the laws L (or in more colorful, if misleading, language, the models that "obey" the laws L)," (p4, 2002). Hence, any theory whose domain extends to the entire universe, (i.e. any cosmological theory), has a multiverse

associated with it: namely, the class of all models of that theory. For a set Σ of sentences, let Mod Σ denote the class of all models of Σ. The class Mod Σ, if non–empty, is too large to be a set, (Enderton 2001, p92). Hence, in this sense, the multiverse associated with any cosmological theory is too large to be a set itself. At face value, this seems to contradict the fact that, whilst many of the multiverses considered in cosmology possess an infinite cardinality, and may even form an infinite–dimensional space, they are, nevertheless, sets. This apparent contradiction arises because physicists tacitly restrict the range of interpretations of the languages in which their theories are expressed, holding the meaning of the predicates fixed, but allowing the variables to range over different domains.

A theory T is complete if for any sentence σ, either σ or $\neg\sigma$ belongs to T. If a sentence is true in some models of a theory but not in others, then the theory is incomplete. Gödel's first incompleteness theorem demonstrates that Peano arithmetic is incomplete. Hence, any theory which includes Peano arithmetic will also be incomplete. If a final theory of everything includes Peano arithmetic, (an apparently moderate requirement), then the final theory will be incomplete. Such a final theory of everything would not eliminate contingency. There would be sentences true in some models of a final theory, but not true in others. Hence, the multiverse hypothesis looms over even a hypothetical final theory of everything. Gödel's incompleteness theorem dispels the possibility that there is only one logically possible physical universe.

Jesus Mosterin (2004) points out that "the set of all possible worlds is not at all defined with independence from our conceptual schemes and models. If we keep a certain model (with its underlying theories and mathematics) fixed, the set of the combinations of admissible values for its free parameters gives us the set of all possible worlds (relative to that model). It changes every time we introduce a new cosmological model (and we are introducing them all the time). Of course, one could propose considering the

set of all possible worlds relative to all possible models formulated in all possible languages on the basis of all possible mathematics and all possible underlying theories, but such consideration would produce more dizziness than enlightment."

Mosterin's point here is aimed at the anthropic principle, and the suggestion that there are multiverses which realize all possible combinations of values for the free parameters in physical theories such as the standard model of particle physics. At face value, these are different types of multiverse than the ones proposed in this paper, which are obtained by varying mathematical structures, and by taking all the models of a fixed mathematical structure, rather than by taking all the values of the free parameters within a theory. However, the values chosen for the free parameters of a theory correspond to a choice of model in various parts of the theory. For example, the free parameters of the standard model of particle physics include the coupling constants of the strong and electromagnetic forces, two parameters which determine the Higgs field potential, the Weinberg angle, the masses of the elementary quarks and leptons, and the values of four parameters in the Kobayashi–Maskawa matrix which specifies the 'mixing' of the $\{d, s, b\}$ quark flavours in weak force interactions. The value chosen for the coupling constant of a gauge field with gauge group G corresponds to a choice of metric in the lie algebra $\mathfrak{g}$, (Derdzinksi 1992, p114–115); the Weinberg angle corresponds to a choice of metric in the lie algebra of the electroweak force, (ibid., p104–111); the values chosen for the masses of the elementary quarks and leptons correspond to the choice of a finite family of irreducible unitary representations of the local space–time symmetry group, from a continuous infinity of alternatives on offer; and the choice of a specific Kobayashi–Maskawa matrix corresponds to the selection of a specific orthogonal decomposition $\sigma_{d'} \oplus \sigma_{s'} \oplus \sigma_{b'}$ of the fibre bundle which represents a generalization of the $\{d, s, b\}$ quark flavours, (ibid., p160).

In general relativity, a universe is represented by a 4–dimensional differential manifold $\mathcal{M}$ equipped with a metric tensor field g and a set of matter fields and gauge force fields $\{\phi_i\}$ which generate an energy–stress–momentum tensor T that satisfies the Einstein field equations

$$T = 1/(8\pi G)(\mathrm{Ric} - 1/2\ \mathrm{S}\, g)\,.$$

Ric denotes the Ricci tensor field determined by g, and S denotes the curvature scalar field. The matter fields have distinctive equations of state, and include fluids, scalar fields, tensor fields, and spinor fields. Gauge force fields, such as electromagnetism, are described by n–form fields. Hence, one can define a general relativistic multiverse to be the class of all models of such n–tuples $\{\mathcal{M}, g, \phi_1, ...\}$, interpreted in this restricted sense.

Alternatively, to take an example suggested by David Wallace (2001), quantum field theory represents a universe to be a Lorentzian manifold $(\mathcal{M}, g)$ which is equipped with a Hilbert space $\mathscr{H}$, a density operator ρ on $\mathscr{H}$, and a collection of operator–valued distributions $\{\hat{\phi}_i\}$ on $\mathcal{M}$ which take their values as bounded self–adjoint operators on $\mathscr{H}$. A quantum field theory multiverse is the class of all models of such n–tuples $\{\mathcal{M}, g, \mathscr{H}, \rho, \hat{\phi}_1, ...\}$, interpreted in this restricted sense.

Assuming that cosmological theories are axiomatizable,[2] one can abstract from the theories which may just apply to our universe or a collection of universes in a neighbourhood of our own, by varying or relaxing the axioms. Each set of axioms has its own class of models, which, in this context, provides a multiverse. Hence, by varying or relaxing the axioms of a theory which is empirically verified in our universe, one generates a tree–like hierarchy of multiverses, some of which are sibling to the original class of models, and some of which are parents or 'ancestors' to the original class.

[2]I.e. assuming that there is a decidable set of sentences (Enderton p62) in such a theory, from which all the sentences of the theory are logically implied.

If an empirically verified theory, with a class of models $\mathfrak{M}$, and a set of laws L, defines a subclass of models $\mathfrak{M}_L$, and if those laws contain a set of free parameters $\{p_i : i = 1, ..., n\}$, then one has a different class of models $\mathfrak{M}_{L(p_i)}$ for each set of combined values of the parameters $\{p_i\}$. These multiverses are sibling to each other in the hierarchy. By relaxing the requirement that any set of laws be satisfied in the mathematical structure in question, one obtains a multiverse $\mathfrak{M}$ which is parent to all the multiverses $\mathfrak{M}_{L(p_i)}$. By varying the axioms that define the original mathematical structure, one obtains sibling mathematical structures which have classes of models $\{\mathfrak{N}_j\}$ sibling to $\mathfrak{M}$. By relaxing the axioms that define the original mathematical structure, one steadily obtains more general mathematical structures which have more general classes of model $\mathfrak{P}_k \supset \mathfrak{M}$. Each such $\mathfrak{P}_k$ is a parent or ancestor in the hierarchy to all the multiverses $\{\mathfrak{N}_j\}$. For example, if we take general relativistic cosmology to provide a theory of our own universe, one can obtain a selective hierarchy of multiverses such as the following:

- All Friedmann–Robertson–Walker (FRW) spatially isotropic Lorentzian 4–manifolds and matter field pairings.
- All Bianchi spatially homogeneous Lorentzian 4–manifolds and matter field pairings.
- All Lorentzian 4–manifolds which solve the canonical/initial–value formulation of the Einstein field equations with respect to some combination of matter fields and gauge fields,[3] for a fixed spatial topology Σ.
- All Lorentzian 4–manifolds which satisfy the Einstein field equations with respect to some combination of matter fields and gauge fields, for the value of the gravitational constant $G \approx 6.67 \times 10^{-8} \mathrm{cm}^3 g^{-1} s^{-2}$ that we observe in our universe.

[3]Note that matter fields and gauge force fields must satisfy their own constraint equations and evolution equations.

- All Lorentzian 4–manifolds which satisfy the Einstein field equations with respect to some combination of matter fields and gauge fields, for a different value of the gravitational constant G.
- All Lorentzian 4–manifolds equipped with some combination of matter fields and gauge fields, irrespective of whether they satisfy the Einstein field equations.
- All Lorentzian 4–manifolds.
- All manifolds of arbitrary dimension and geometrical signature, which satisfy the Einstein field equations with respect to some combination of matter fields and gauge fields.
- All 4–manifolds equipped with combinations of smooth tensor and spinor fields[4] which satisfy differential equations.
- All 4–manifolds.
- All differential manifolds of any dimension equipped with combinations of smooth tensor and spinor fields which satisfy differential equations.
- All differential manifolds.
- All topological manifolds equipped with combinations of continuous functions which satisfy algebraic[5] equations.
- All topological spaces with the cardinality of the continuum equipped with combinations of continuous functions which satisfy algebraic equations.

[4]Note that *global* spinor fields require a manifold to possess a spin structure.

[5]Algebraic equations involve only the operations defined upon the algebra of functions $\mathscr{A}$. This means operations such as scalar multiplication, sum, product, and derivation. In this context, a derivation of $\mathscr{A}$ is a mapping $X\colon \mathscr{A} \to \mathscr{A}$ which is linear, $X(af + bg) = aXf + bXg$, and which satisfies the so–called Leibniz rule, $X(fg) = fX(g) + X(f)g$.

- All topological spaces of any cardinality equipped with combinations of continuous functions which satisfy algebraic equations.
- All topological spaces.
- All sets equipped with combinations of functions which satisfy algebraic equations.
- All sets.
- All categories.

Mosterin comments acerbically that "authors fond of many universes talk about them in a variety of incompatible ways. The totality of the many universes accepted by an author forms the multiverse for that author. There are at least as many multiverses as authors talking about them; in fact, there are more, as some authors have several multiverses to offer," (2004). Bearing this in mind, it should be declared that this paper is only interested in multiverses consisting of the models of empirically verified theories, or generalisations obtained from such theories. Hence, multiverses derived from supersymmetry, supergravity, superstring or M theory, will not be considered.

From the list of multiverses above, we now proceed to consider a couple of interesting cases. In particular, we will be interested in understanding how special or typical our own universe is with respect to these multiverses.

2. THE MULTIVERSE OF SPATIALLY HOMOGENEOUS MODELS

The purpose of this section is to analyse an argument by Collins and Hawking that our own spatially isotropic universe is extremely atypical even in the space of spatially homogeneous models. To gain some understanding of the argument, however,

we need to begin with some facts about spatially homogeneous models.

It is generally believed that, up to local isometry, a spatially homogeneous model, equipped with a fluid satisfying a specific equation of state, can be uniquely identified by specifying both its Bianchi type, and by specifying its dynamical history. The Bianchi type classifies the 3–dimensional Lie algebra of Killing vector fields on the spatially homogeneous hypersurfaces of such a model.

In any 3–dimensional Lie algebra, the space of all possible bases is 9–dimensional, and the general linear group $GL(3,\mathbb{R})$ acts simply transitively upon this space of bases. Hence, if one fixes a basis, then one can establish a one–to–one mapping between the bases in the Lie algebra and the matrices in $GL(3,\mathbb{R})$. Needless to say, $GL(3,\mathbb{R})$ is a 9–dimensional group. Now, specifying the structure constants of a 3–dimensional Lie algebra relative to a particular basis uniquely identifies a particular Bianchi type. Given a Bianchi type fixed in such a manner, one can allow $GL(3,\mathbb{R})$ to act upon the space of bases in the Lie algebra. Under some changes of basis, the structure constants will change, whilst under other changes of basis, they will remain unchanged. Hence, the action of $GL(3,\mathbb{R})$ upon the space of structure constants for a particular Bianchi type is multiply transitive; the action of $GL(3,\mathbb{R})$ upon the space of structure constants has a non–trivial stability subgroup. The dimension of this stability subgroup depends upon the Bianchi type.

Another way of looking at this is to consider the 6–dimensional space of all possible structure constants for all possible 3–dimensional Lie algebras. The general linear group $GL(3,\mathbb{R})$ acts upon this space. The $GL(3,\mathbb{R})$–action does not map the structure constants of one Bianchi type into another; changing the basis in a Lie algebra will not give you the structure constants of a different Lie algebra. Hence, each orbit of $GL(3,\mathbb{R})$ in the space of all structure constants corresponds to a particular Bianchi type. The

dimension p of each orbit is given in the table below, (Collins and Hawking 1973, p321; Hewitt *et al* 1997, p210; MacCallum 1979, p541). The dimension of the $GL(3,\mathbb{R})$–orbit for each Bianchi type equals the number of free parameters required to specify the structure constants for that Bianchi type. Given that $GL(3,\mathbb{R})$ is a 9–dimensional group, it follows that its stability group at each point in the space of structure constants will be $9-p$ dimensional. This is the dimension of the space of bases under which the structure constants are unchanged.

Table 1: Dimension of the $GL(3,\mathbb{R})$–orbits for each Bianchi type.

Type	Dimension
I	0
II	3
VI_0	5
VII_0	5
VIII	6
IX	6
V	3
IV	5
VI_h	6
VII_h	6

Collins and Hawking (1973) argued that the set of spatially homogeneous but anisotropic models which tend towards isotropy as $t \to \infty$ is of measure zero in the set of all spatially homogeneous initial data. They first excluded all the Bianchi types whose metrics are of measure zero in the space of all 3–dimensional homogeneous metrics. This includes all the Bianchi types whose $GL(3,\mathbb{R})$–orbits are of dimension less than 6. They then excluded types VI_h and VIII on the grounds that they do not contain any FRW models as limiting cases. However, the class of Bianchi type

VII_h models contain the ever–expanding FRW universes, with spatial curvature $k < 0$, as special cases, and the class of type IX models contain the closed $k > 1$ FRW universes. In the class of type VII_h models, where σ_{ij} is the shear tensor and H is a generalized Hubble parameter, approach to isotropy was defined to mean that the 'distortion' $\sigma/H \to 0$ as $t \to \infty$, and that the cumulative distortion $\int^t \sigma \, dt$ approaches a constant as $t \to \infty$, where $\sigma = \sigma^i{}_i$. Collins and Hawking concluded that there is no open neighbourhood of the FRW models in the space of either type VII_h or type IX metrics which tends towards isotropy. However, they did find that in the space of type VII_0 metrics, which contains the $k = 0$ FRW universes as special cases, and with the matter assumed to be zero–pressure 'dust', there is an open neighbourhood about such FRW universes which do approach isotropy. However, the space of type VII_0 metrics is of measure zero in the space of all homogeneous metrics. Moreover, the assumption of a zero–pressure matter field prevents one applying this result to universes which have a radiation–dominated phase, such as our own is believed to have undergone in its early history.

From their conclusion that the set of spatially homogeneous models which approach isotropy as $t \to \infty$ is of measure zero, Collins and Hawking also inferred that isotropic models are unstable under spatially homogeneous perturbations. In other words, it was argued that a model which is initially almost isotropic and spatially homogeneous, will tend towards anisotropy. However, Barrow and Tipler correctly point out that the requirement $\sigma/H \to 0$ is the condition of 'asymptotic stability', and the open FRW universe in the type VII_h Bianchi class is stable under spatially homogeneous perturbations in the sense that σ/H approaches a constant. According to Barrow and Tipler this shows "that isotropic open universes are stable in the same sense that our solar system is stable. As $t \to \infty$ there exist spatially homogeneous perturbations with $\sigma/H \to$ constant but there are none with $\sigma/H \to \infty$. The demand for asymptotic stability is too

strong a requirement," (1986, p425). Whilst the set of spatially homogeneous models which approach isotropy is of measure zero, our own universe is not exactly isotropic, it is merely 'almost–isotropic', and, moreover, it is growing increasingly anisotropic as a function of time. Collins and Hawking fail to demonstrate that almost–isotropic spatially homogeneous models are of measure zero in the space of spatially homogeneous models. Moreover, our own universe is almost a FRW model in the sense that it has approximate spatial homogeneity and approximate spatial isotropy. Our own universe, then, has been perturbed by inhomogeneous perturbations. Collins and Hawking fail to demonstrate that almost–isotropic and almost–homogeneous universes, like our own, are of measure zero in the space of almost homogeneous models.

To derive their conclusion that type VII_h models fail to isotropize, Collins and Hawking assumed various reasonable conditions defining the nature of matter, but they also assumed a zero cosmological constant, an assumption which recent astronomical evidence for an accelerating universe has cast into serious doubt. Wald (1983) demonstrated that all initially expanding non–type IX Bianchi models with spatial curvature $k \leq 0$ and a positive cosmological constant, do in fact isotropize, expanding exponentially, and tending toward de Sitter space–time, with $\sigma/H \to 0$ as $t \to \infty$. However, Wald's result only applies to Bianchi type IX models if one assumes that the cosmological constant is initially large in comparison with the scalar curvature of the hypersurfaces of homogeneity, (Earman and Mosterin, 1999). In addition, Wald's result has not been extended to spatially inhomogeneous universes. The attempt by Jensen and Stein–Schabes (1987) to extend Wald's result makes the physically unacceptable assumption that the scalar curvature is non–positive throughout the space–time. Even with initial non–positive scalar curvature, in a spatially inhomogeneous universe one would expect pockets of positive curvature to form with the passage of time, such as those

around black holes, stars and planets. The exterior Schwarzschild solution, generalized to the case of a positive cosmological constant, has such positive curvature, and doesn't evolve towards de Sitter space–time, (Earman and Mosterin 1999). It remains an open question whether initially expanding, spatially inhomogeneous universes of initial non–positive curvature, which develop localized pockets of positive curvature, evolve towards de Sitter space–time if they have a positive cosmological constant.

It is true, however, that the finite–dimensional space of exact FRW models is of measure zero in the finite–dimensional space of spatially homogeneous models, and Collins and Hawking demonstrate that spatially homogeneous anisotropic models do not generally approach isotropic ones, unless the cosmological constant is non–zero. As we will see, the space of spatially homogeneous models is itself the complement of an open dense subset in the infinite–dimensional space of all solutions to the Einstein field equations. Hence, the space of exact FRW models is as numerous in the space of spatially homogeneous models as the integers are in the set of real numbers, and the set of spatially homogeneous models is as prevalent in the space of all solutions to the Einstein field equations as the points in the surface of solid ball are to the points in the interior of the solid ball.

3. THE MULTIVERSE OF ALL SOLUTIONS TO THE EINSTEIN FIELD EQUATIONS

For each 4–dimensional manifold $\mathcal{M}$, one has the space of all solutions to the Einstein field equations on that manifold, $\mathscr{E}(\mathcal{M})$. Rather than dealing with matter field solutions, existing analysis has concentrated on vacuum solutions of the Einstein field equations, and yet further restrictions have been placed on the topology and geometry of the solutions considered. For example, work has been done on the space $\tilde{\mathscr{E}}(\mathcal{M})$ of vacuum solutions of a globally hyperbolic space–time with compact spatial topology

Σ, containing a constant mean extrinsic curvature Cauchy hypersurface. The restrictions placed upon such solution spaces means that they really just contain the solutions of initial data sets. Nothing is revealed about spaces of solutions which do not admit an initial–value formulation.

$\tilde{\mathscr{E}}(\mathcal{M})$ is not a manifold, but a stratified space. The points representing solutions with a non–trivial isometry group are said to have conical neighbourhoods. The strata consist of space–times with conjugate isometry groups, (Isenberg and Marsden 1982, p187).

Two space–time solutions are considered to be physically equivalent if one is isometric to the other, hence work has been done on the quotient space $\tilde{\mathscr{E}}(\mathcal{M})/\mathscr{D}(\mathcal{M})$ with respect to the diffeomorphism group $\mathscr{D}(\mathcal{M})$ of the 4–manifold. Again, $\tilde{\mathscr{E}}(\mathcal{M})/\mathscr{D}(\mathcal{M})$ is not a manifold itself, but a stratified space. The points in $\tilde{\mathscr{E}}(\mathcal{M})/\mathscr{D}(\mathcal{M})$ with no isometry group form an open and dense subset. In other words, the stratum of equivalence classes of solutions to Einstein's equations with no isometries, is open and dense in $\tilde{\mathscr{E}}(\mathcal{M})/\mathscr{D}(\mathcal{M})$, (Isenberg and Marsden, p210). This is thought to confirm the general presumption that space–times with no symmetry are extremely typical in the set of space–times, and space–times with some degree of symmetry are extremely special.[6] The Friedmann–Robertson–Walker models, and perturbations thereof, considered to be the physically realistic models for our universe, are believed to be extremely atypical in the space of all solutions to the Einstein field equations. As Turner comments, "even the class of slightly lumpy FRW solutions occupies only a set of measure zero in the space of initial data," (Turner 2001, p655).

This, however, might be a somewhat hasty conclusion to reach. As Ellis comments: "We have at present no fully satisfactory measure of the distance between two cosmological models... or of the

[6]In this context, the term 'special' will be considered equivalent to the term 'atypical'.

probability of any particular model occurring in the space of all cosmologies. Without such a solid base, intuitive measures are often used... the results obtained are dependent on the variables chosen, and could be misleading — one can change them by changing the variables used or the associated assumptions. So if one wishes to talk about the probability of the universe or of specific cosmological models, as physicists wish to do, the proper foundation for those concepts is not yet in place," (1999).

When a collection of objects forms a finite–dimensional manifold, and in the absence of any sort of probability measure, one uses the Lebesgue measure, or an analogue thereof, to define precisely what it means for a property to be typical or special. If the manifold provides a finite measure space, then a property which is only possessed by a subset of measure zero is considered to be special, while if the manifold provides an infinite measure space, then a property which is only possessed by a subset of finite measure is considered to be special. In both cases, the property which defines the complement is considered to be typical of the collection of objects. In the case of an infinite–dimensional topological vector space, there is no finite, translation invariant measure to provide an analogue of the Lebesgue measure, and although non–translation invariant Gaussian measures do exist on such infinite–dimensional vector spaces, in the case of an infinite–dimensional manifold there is no diffeomorphism–invariant measure which is considered to be suitable. As Callender comments, "debates about likely versus unlikely initial conditions without a well–defined probability are just intuition–mongering," (2004). Given the difficulties with finding such a measure, topological notions of typical and special have been proposed to replace the measure–theoretic notions.

The first candidate for a topological notion of typicality is the notion of a dense subset. However, the set of rational numbers is dense in the set of real numbers, despite having a lower cardinality, and despite being of Lebesgue measure zero, hence a dense subset

of a topological space is not necessarily considered to be typical. Instead, following Baire, an open and dense subset of a topological space is 'strongly typical', and a set which contains the intersection of a countable collection of dense and open subsets is 'residual', or 'typical' (Heller 1992, p72). The irrational numbers are a residual subset of the reals, and therefore typical, and their complement, the rational numbers, are atypical. Baire's theorem shows that in a complete metric space or a locally compact Hausdorff space, a countable intersection of open dense subsets must itself be dense. However, the attraction of the Baire definition of typicality is mitigated by the fact that sets which are open and dense in $\mathbb{R}^n$ can have arbitrarily small Lebesgue measure, (Hunt *et al* 1992). Moreover, an infinite–dimensional manifold fails to be locally compact. It is therefore far from clear that a collection of universes which is open and dense in a multiverse collection, should be considered as typical, or that points which belong to the complement, (those with some degree of symmetry), should be considered special. There is no *a priori* notion of typicality on such sets, so it is rather unwise to make such presumptions. In the classical statistical mechanics of gases, it is always asserted that a homogeneous distribution of the gas is the macrostate of highest entropy because it has the greatest volume of microstates, the greatest volume of phase space, associated with it. This means that the homogeneous macrostates must have the highest measure, and certainly not measure zero. Thus, there is certainly no *a priori* reason from the finite–dimensional case to think that in the case of continuous fields, a highly symmetrical configuration should be atypical.

Note that the present universe only approximates a FRW model on length scales greater than 100 Mpc. On smaller length scales, the universe exhibits large inhomogeneities and anisotropies. The distribution of matter is characterised by walls, filaments and voids up to 100 Mpc, with large peculiar velocities relative to the rest frame defined by the cosmic microwave back-

ground radiation (CMBR). Whilst the CMBR indicates that the matter in the universe was spatially isotropic and homogeneous to a high degree when the universe was $10^4 - 10^5$ yrs old, the distribution and motion of galaxies is an indicator of the distribution of matter in the present era, when the universe is $\sim 10^{10}$ yrs old. Due to the tendency of gravitation to amplify small initial inhomogeneities, the level of inhomogeneity in the distribution of matter has been growing as a function of time.

In contrast with the behaviour of a gas in classical statistical mechanics, the distribution of matter in a gravitational system will become more 'clumpy' as a function of time. Hence, to preserve consistency with the second law of thermodynamics it is generally suggested that for a gravitational system, such clumpy configurations correspond to higher–entropy macrostates than configurations with a uniform distribution of matter. Barrow and Tipler, for example, suggest that a gravitational entropy would measure the deviation of a universe from exact spatial isotropy and homogeneity (1986, p446). Penrose suggests that gravitational entropy is related to the Weyl tensor $C_{\alpha\beta\gamma\delta}$, which is zero for exact FRW models, but non–zero for the space–time around a massive body such as a star or black hole. However, anisotropy does not entail a non–zero Weyl tensor, and a non–zero Weyl tensor does not entail inhomogeneity. Barrow and Tipler note that 'many' space–times tend towards a plane gravitational wave geometry, which has a zero Weyl tensor irrespective of the level of anisotropy (1986, p447). On the other hand, there are many anisotropic but spatially homogeneous models, in which the Weyl tensor is non–zero. Thus, if we accept that the Weyl tensor measures gravitational entropy, then even a universe which is exactly spatially homogeneous, will, if it exhibits anisotropic shears and a non–zero Weyl tensor, possess a higher entropy than a perfectly isotropic and homogeneous FRW universe.

Ellis (2002) suggests that the spatial divergence of the electric part of the Weyl tensor $E_{\alpha\gamma} = C_{\alpha\beta\gamma\delta}U^\beta U^\delta$, for a timelike ob-

server vector field U, may be a measure of gravitational entropy. This quantity is attractive because it does seem to have some correlation to the level of spatial inhomogeneity. One could take the integral $\int_\Sigma ||\nabla_a \rho||\ d\mu$ of the spacelike gradient of the matter density field ρ as a measure of the amount of inhomogeneity in the matter field over a spacelike hypersurface Σ. In the case of a homogeneous matter field, this integral vanishes. Ellis suggests that in the linearized formulation of general relativity,

$$\nabla^a E_{ab} = \frac{1}{3}\nabla_b \rho\ ,$$

hence the integral $\int_\Sigma ||\nabla^a E_{ab}||\ d\ \mu$ may, in some circumstances, be proportional to the level of inhomogeneity, and thence a good indicator of gravitational entropy. Suppose that one takes the integral of $\nabla^a E_{ab}$ for an arbitrary spatially homogeneous but anisotropic model. Being spatially homogeneous, the spatial gradient of ρ vanishes. If one introduces an inhomogeneous perturbation to this model, then the integral of $\nabla^a E_{ab}$ should presumably increase. The inhomogeneity of our own universe is currently growing, and, if the inhomogeneous geometries are far more prevalent, in some sense, than homogeneous geometries, then our universe must be moving into phase space macrostates of much greater entropy.

There is, however, a problem with the notion that gravitational entropy should measure the deviation of a universe from exact isotropy and homogeneity. If there is a positive cosmological constant, and if Wald's theorem does apply to the case of an inhomogeneous universe, then the universe will isotropize, and isotropy requires homogeneity. Even in the absence of a cosmological constant, all gravitational systems, perhaps even black holes, will eventually 'evaporate', and this might also entail a return to homogeneity. Thus, if the universe tends towards either isotropy or homogeneity in the long–term, perhaps gravitational entropy should not measure the deviation of a universe from exact isotropy and homogeneity.

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SUMMARY

POSSIBLE PHYSICAL UNIVERSES

The purpose of this paper is to discuss the various types of physical universe which could exist according to modern mathematical physics. The paper begins with an introduction that approaches the question from the viewpoint of ontic structural realism. Section 2 takes the case of the 'multiverse' of spatially homogeneous universes, and analyses the famous Collins–Hawking argument, which purports to show that our own universe is a very special member of this collection. Section 3 considers the multiverse of all solutions to the Einstein field equations, and continues the discussion of whether the notions of special and typical can be defined within such a collection.

STRESZCZENIE

MOŻLIWE FIZYCZNE WSZECHŚWIATY

Celem artykułu jest analiza różnych typów fizycznych wszechświatów, które mogą istnieć według współczesnej fizyki matematycznej. We wstępie przedstawiono tę problematykę z punktu widzenia ontycznego realizmu strukturalistycznego. W podrozdziale 2 rozważa się „wieloświat" przestrzennie jednorodnych wszechświatów i poddaje analizie słynny argument Collinsa–Hawkinga, którego celem było wykazanie, że nasz wszechświat jest bardzo szczególnym elementem tej rodziny. W podrozdziale 3 omawia się wieloświat wszystkich rozwiązań Einsteinowskich równań pola i rozważa, czy pojęcia „szczególny" i „typowy" mogą być sensownie zdefiniowane w tej rodzinie.